Science and Engineering Applications 2(1) (2017) 96-108 ISSN-2456-2793(Online)

Contents lists available at JFIPS

Science and Engineering Applications

Journal home page: JFIPS

SAEA

Homotopy Perturbation for Excited Nonlinear Equations

Yusry O.El-Dib
Department of Mathematics, Faculty of Education,
Ain Shams University, Roxy, Egypt
*Email: yusryeldib52@hotmail.com

Abstract

The purpose of this paper is to apply a version of homotopy technique to excited nonlinear problems. A modulation for the homotopy perturbation is introduced in order to be successfully for nonlinear equations having periodic coefficients. The nonlinear damping Mathieu equation has been studied as a simplest example. The analysis proceeds without assuming weakly nonlinearity and without presence of small factor for the periodic term. In this analysis, two nonlinear solvability conditions are imposed. One of them imposed in the first-order homotopy perturbation and used to study the stability behavior at resonance and non-resonance cases. The second level of the perturbation produces another solvability condition and used to determine the unknowns appear in solution for the first-order solvability condition. The method can be, also, used for excited linear equation. Stability conditions, and also the transition curves, are formulated independent of the small parameter i.e. in the unperturbed form as an alternative to classical methods.

Keywords: Homotopy perturbation method, nonlinear Mathieu equation, modulation method, parametric resonance.

Received : 21/02/2017 Published online : 5/4/2017

1. Introduction

In [1], perturbation methods depend on small parameter and choose unsuitable small parameter can be lead to wrong solution. Homotopy is an important part of topology [2] and it can convert any non-linear problem in to a finite linear problems and it doesn't depend on small parameter.

The homotopy perturbation method, first proposed by Ji Huan He [3,4,5,6], has successfully been applied to solve many types of linear and nonlinear functional equations. This method, which is a combination of homotopy in topology and classic perturbation techniques, provides us with a convenient way to obtain analytic or approximate solutions to a wide variety of problems arising in different fields. He used the homotopy perturbation method to solve Lighthill equation [3], Duffing equation [7] and Blasius equation [8], then the idea go through and has been used to solve nonlinear wave equations [9], boundary value problems [10]. E. Babolian, J. Saeidian and A. Azizi [11] apply homotopy perturbation method to Burgers equation, the regularized long wave equation and the modified Korteweg-de Vries equation. M. Rabbani [12], introduce a new homotopy perturbation method for solving non-linear problems through projection method. A new homotopy perturbation method for solving linear and nonlinear Schrödinger equations has been addressed by Z.Ayati1, J. Biazar and S. Ebrahimi [13].

In this work, we apply the homotopy perturbation method to linear or nonlinear equations having periodic coefficients such

Mathieu equation which has been of great importance among researchers. The Mathieu equation serves as a useful model for many interesting problems in applied mathematics, engineering mechanics, etc.

2. Mathematical formulation

To explain the proposed technique, consider the following damping cubic nonlinear Mathieu equation as an illustrative example:

$$\frac{d^2y}{dt^2}+\mu\frac{dy}{dt}+(\omega^2+2q\cos 2\Omega t)y=ky^3. \tag{1}$$

This equation is without any restrictions on its coefficients. The homotopy perturbation method can be considered as combination of the classical perturbation technique and the homotopy (whose origin is in the topology [2]), but not restricted to the limitations of traditional perturbation methods. For example, this method does require neither small parameter nor linearization, and only requires little iteration to obtain accurate solutions [3] and [4].

We define the two parts of equation (1) as $L(y)$ and $N(y)$, where

$$L(y)=\frac{d^2y}{dt^2}+\omega^2 y,\ \text{and}\quad N(y)=\mu\frac{dy}{dt}+2qy\cos 2\Omega t-ky^3. \tag{2}$$

Construct the homotopy statement as

$$H(y,\rho)=L(y)+\rho N(y)=0;\quad \rho\in[0,1] \tag{3}$$

As in He's homotopy perturbation method, it is obvious that when $\rho=0$, Eq. (3) becomes the harmonic equation; $L(y)=0$. Thus,

$$\frac{d^2y(t)}{dt^2}+\omega^2 y(t)=0. \tag{4}$$

According to linear differential equations theory, the general solution of (4) is expressed in terms of two linearly independent solutions, say, $e^{i\omega t}$ and $e^{-i\omega t}$. Thus, the composite solutions may be in the form

$$y(t)=Ae^{i\omega t}+\bar{A}e^{-i\omega t}, \tag{5}$$

where A and its complex conjugate $\bar{A}$ are arbitrary constants of integration. Eq. (3) becomes the original nonlinear Mathieu equation (1) as $\rho=1$. For arbitrary the small parameter ρ, the solution of equation (3) can be sought in terms of ρ so that the function $y(t)$ becomes $y(t,\rho)$. Accordingly, equation (3) can be rewrite as

$$H(y,\rho)=\left(\frac{d^2}{dt^2}+\omega^2\right)y(t,\rho)+\rho\left(\mu\frac{d}{dt}+2q\cos 2\Omega t-ky^2(t,\rho)\right)y(t,\rho)=0. \tag{6}$$

It can be noticed that the homotopy function (6) is essentially the same as (3), except for function $y(t,\rho)$, which contain embedded the homotopy parameter ρ. The introduction of that parameter within the differential equation is a strategy to redistribute the periodic part between the successive iterations of the homotopy method, and thus increase the probabilities of finding the sought solution. Thus, as ρ moves from 0 to 1, the function $y(t,\rho)$ moves from $y_0(t)$ to $y_{app}(t)$. Expand the function $y(t,\rho)$ as a power series in the small parameter ρ such that

$$y(t,\rho) = y_0(t) + \rho y_1(t) + \rho^2 y_2(t) + \ldots \tag{7}$$

where $y_n(t); n = 1,2,3,\ldots$ are unknowns in needs to determined. Substituting this expansion into the homotopy equation (6) and equating terms with identical powers of ρ, leads to

$$\rho^0 : \frac{d^2 y_0}{dt^2} + \omega^2 y_0 = 0, \tag{8}$$

$$\rho^1 : \frac{d^2 y_1}{dt^2} + \omega^2 y_1 = -\mu \frac{dy_0}{dt} - 2q\cos 2\Omega t \, y_0 + k y_0^3, \tag{9}$$

$$\rho^2 : \frac{d^2 y_2}{dt^2} + \omega^2 y_2 = -\mu \frac{dy_1}{dt} - 2q\cos 2\Omega t \, y_1 + 3k y_0^2 y_1, \ldots \tag{10}$$

Equation (8) can be satisfied by

$$y_0(t) = A e^{i\omega t} + \overline{A} e^{-i\omega t}. \tag{11}$$

Substituting (11) into equation (9) gets the for

$$\begin{aligned} \frac{d^2 y_1}{dt^2} + \omega^2 y_1 = &-i\omega\mu\left(A e^{i\omega t} - \overline{A} e^{-i\omega t}\right) - qA\left(e^{i(\omega+2\Omega)t} + e^{i(\omega-2\Omega)t}\right) - q\overline{A}\left(e^{-i(\omega+2\Omega)t} + e^{-i(\omega-2\Omega)t}\right) \\ &+ k\left(A^3 e^{3i\omega t} + 3A^2\overline{A} e^{i\omega t} + 3\overline{A}^2 A e^{-i\omega t} + \overline{A}^3 e^{-3i\omega t}\right) \end{aligned} \tag{12}$$

Before analyzed the first-order problem we must distinguish between two cases. The case of the frequency Ω is not equal the nature frequency ω (which is known as the non-resonance case). The second one is the specific case when Ω approaches ω (which is known as the resonance case).

For arbitrary frequency Ω, there are secular terms appears in equation (12). Elimination such secular terms requires that the arbitrary constant A be zero. This means that expansion (7) cannot be successfully to obtain valid solution for excited homotopy equation (6).

3. The modulation procedure

To obtain uniform expansions for problems of this kind, the expansion (7) needs to be modified. If we modulate the initial solution (5) so that the constant A becomes $A(\tau)$ with $\tau = \rho t$, such that

$$\frac{dA}{dt} = \rho \frac{dA}{d\tau} \quad \text{and} \quad \frac{d^2 A}{dt^2} = \rho^2 \frac{dA}{d\tau^2}. \tag{13}$$

Then (11) in the modulate case becomes

$$Y_0(t,\tau) = A(\tau) u_0(t) + \overline{A}(\tau) \overline{u}_0(t), \tag{14}$$

where

$$u_0(t) = e^{i\omega t} \quad \text{and} \quad \overline{u}_0(t) = e^{-i\omega t}. \tag{15}$$

Consequently, the homotopy state, equation (6), in the modulated form becomes

$$\left(\frac{d^2}{dt^2} + \omega^2\right) Y(t,\tau,\rho) + \rho\left(\mu \frac{d}{dt} + 2q\cos 2\Omega t\right) Y(t,\tau,\rho) = \rho k Y^3(t,\tau,\rho). \tag{16}$$

It is convenient to choose the modulated function $Y(t,\tau,\rho)$ in separated variables as

$$Y(t,\tau,\rho)=A(\tau)u(t,\rho)+\bar{A}(\tau)\bar{u}(t,\rho). \tag{17}$$

The function $u(t,\rho)$ can be expanded as a power series in the small parameter ρ such that

$$u(t,\rho)=u_0(t)+\rho u_1(t)+\rho^2 u_2(t)+... \tag{18}$$

where $u_n(t); n=1,2,3,...$ are unknowns to be evaluated. If the expansion (18) is substituted into (17) then gets

$$\begin{aligned} Y(t,\tau,\rho) &= A(\tau)(u_0(t)+\rho u_1(t)+\rho^2 u_2(t)+...)+cc. \\ &= Y_0(t,\tau)+\rho Y_1(t,\tau)+\rho^2 Y_2(t,\tau)+... \end{aligned} \tag{19}$$

where $cc.$ indicates to the complex conjugate for the preceding terms and

$$Y_n(t,\tau)=A(\tau)u_n(t)+\bar{A}(\tau)\bar{u}_n(t). \tag{20}$$

It is noted that

$$\frac{d}{dt}Y(t,\tau,\rho)=\frac{d}{dt}\left[A(\tau)u(t,\rho)+cc\right]=A(\tau)\dot{u}(t,\rho)+\rho u(t,\rho)A'(\tau)+cc, \tag{21}$$

and $$\frac{d^2}{dt^2}Y(t,\tau,\rho)=A(\tau)\ddot{u}(t,\rho)+2\rho A'(\tau)\dot{u}(t,\rho)+\rho^2 u(t,\rho)A''(\tau)+cc, \tag{22}$$

where dots indicate differentiation with respect to the time t, while dashes refer to the derivative with respect to the time modulate τ. Substituting (17) into equation (16) using (21) and (22) gives

$$\begin{aligned} & A(\ddot{u}+\omega^2 u)+\rho(2\dot{u}A'+\mu A\dot{u}+2qAu\cos 2\Omega t) \\ & +\rho^2(A''+\mu A')u-\rho k(A^3u^3+3A^2\bar{A}u^2\bar{u})+cc.=0. \end{aligned} \tag{23}$$

Equation (23) remains obey the same homotopy concept, because it's become the same harmonic equation (4) as $\rho\to 0$. In addition $\lim_{\rho\to 1} A'=\frac{d}{d\tau}\left(\lim_{\rho\to 1} A\right)=0,$ consequently the original equation (1) is found.

In the light of (18), the modulate homotopy equation (23) will be expanded as a power series in ρ so that the following non-homogenous harmonic equations are imposed:

$$\rho^0: A(\ddot{u}_0+\omega^2 u_0)+\bar{A}(\ddot{\bar{u}}_0+\omega^2\bar{u}_0)=0, \tag{24}$$

$$\rho^1: A(\ddot{u}_1+\omega^2 u_1)+2\dot{u}_0A'+\mu A\dot{u}_0+2qAu_0\cos 2\Omega t-kA^2(Au_0^3+3\bar{A}u_0^2\bar{u}_0)+cc.=0, \tag{25}$$

$$\begin{aligned} \rho^2: & A(\ddot{u}_2+\omega^2 u_2)+2\dot{u}_1A'+\mu A\dot{u}_1+2qAu_1\cos 2\Omega t+u_0A''+\mu A'u_0 \\ & -3kA^2\left[Au_0^2u_1+\bar{A}(u_0^2\bar{u}_1+2u_0\bar{u}_0u_1)\right]+cc.=0. \end{aligned} \tag{26}$$

It is noted that equation (24) has been satisfied by (15) and the zero- order solution for equation (16) as approved in (14). Substituting (15) into (25) becomes

$$A(\ddot{u}_1+\omega^2 u_1)+\left[i\omega(2A'+\mu A)-3kA^2\bar{A}\right]e^{i\omega t}+qA\left(e^{i(\omega+2\Omega)t}+e^{i(\omega-2\Omega)t}\right)-kA^3e^{3i\omega t}+cc.=0. \tag{27}$$

This equation contains secular terms at the non-resonance case and another secular terms when the applied frequency Ω approaches the natural frequency ω.

4. The non-resonance case

The analysis in this case concerned with the arbitrary chosen for the applied frequency Ω, in equation (27). At this stage, secular terms are removed when

$$A' + \tfrac{1}{2}\mu A + \frac{3ik}{2\omega} A^2 \bar{A} = 0, \tag{28}$$

with its complex conjugate one. This leads to obtain the valid function $u_1(t)$ as

$$u_1(t) = \frac{q}{4\Omega}\left(\frac{e^{i(\omega+2\Omega)t}}{(\Omega+\omega)} + \frac{e^{i(\omega-2\Omega)t}}{(\Omega-\omega)}\right) + \frac{kA^2}{8\omega^2} e^{3i\omega t}. \tag{29}$$

Consequently, the solution at the first-order problem is formulated as

$$Y_1(t,\tau) = \frac{qA(\tau)}{4\Omega}\left(\frac{e^{i(\omega+2\Omega)t}}{(\omega+\Omega)} + \frac{e^{i(\omega-2\Omega)t}}{(\omega-\Omega)}\right) + \frac{q\bar{A}(\tau)}{4\Omega}\left(\frac{e^{-i(\omega+2\Omega)t}}{(\omega+\Omega)} + \frac{e^{-i(\omega-2\Omega)t}}{(\omega-\Omega)}\right) + \frac{k}{8\omega^2}\left(A^3(\tau)e^{3i\omega t} + \bar{A}^3(\tau)e^{-3i\omega t}\right) \tag{30}$$

Substituting (15) and (29) into (26), using (28), yields

$$\begin{aligned} A(\ddot{u}_2 + \omega^2 u_2) &+ \left[A'' + \mu A' + \frac{q^2 A}{2(\Omega^2-\omega^2)} - \frac{3k^2}{8\omega^2} A^3 \bar{A}^2\right] e^{i\omega t} + \frac{3k^2 A^4 \bar{A}}{8\omega^2} e^{3i\omega t} - \frac{3k^2 A^5}{8\omega^2} e^{5i\omega t} \\ &+ \frac{q^2 A}{4\Omega(\Omega^2-\omega^2)}\left[(\Omega-\omega)e^{i(\omega+4\Omega)t} + (\Omega+\omega)e^{i(\omega-4\Omega)t}\right] \\ &+ \frac{3kqA^2\bar{A}}{2\omega(\Omega^2-\omega^2)}\left[(\Omega-2\omega)e^{i(\omega+2\Omega)t} - (\Omega+2\omega)e^{i(\omega-2\Omega)t}\right] \\ &- \frac{3kqA^3}{4\Omega(\Omega^2-\omega^2)}\left[(\Omega-\omega)e^{i(3\omega+2\Omega)t} + (\Omega+\omega)e^{i(3\omega-2\Omega)t}\right] + cc. = 0. \end{aligned} \tag{31}$$

The valid solution requires to removing the terms that producing unbounded solution. These terms implies the following nonlinear solvability condition:

$$A'' + \mu A' + \frac{q^2 A}{2(\Omega^2-\omega^2)} - \frac{3k^2}{8\omega^2} A^3 \bar{A}^2 = 0. \tag{32}$$

The second-order solution is found to be

$$\begin{aligned} Y_2(t,\tau) = &\frac{q^2 A(\tau)}{32\Omega^2(\Omega^2-\omega^2)}\left[\frac{(\Omega-\omega)}{(\omega+2\Omega)} e^{i(\omega+4\Omega)t} + \frac{(\Omega+\omega)}{(\omega-2\Omega)} e^{i(\omega-4\Omega)t}\right] \\ &- \frac{3kqA^3(\tau)}{16\Omega(\Omega^2-\omega^2)^2}\left[\frac{(\Omega-\omega)^2}{(2\omega+\Omega)} e^{i(3\omega+2\Omega)t} + \frac{(\Omega+\omega)^2}{(\Omega-2\omega)} e^{i(3\omega-2\Omega)t}\right] \\ &+ \frac{3kqA^2(\tau)\bar{A}(\tau)}{8\Omega^2\omega(\Omega^2-\omega^2)}\left[\frac{(\Omega-2\omega)}{(\omega+\Omega)} e^{i(\omega+2\Omega)t} + \frac{(\Omega+2\omega)}{(\Omega-\omega)} e^{i(\omega-2\Omega)t}\right] \\ &- \frac{k^2 A^5(\tau)}{64\omega^2} e^{5i\omega t} + \frac{3k^2 A^4(\tau)\bar{A}(\tau)}{64\omega^2} e^{3i\omega t} + cc. \end{aligned} \tag{33}$$

If the accuracy to the second-order perturbation is enough, then the approximate solution at the non-resonance case is formulated by substituting (14),(15), (30) and (33) into (19), and setting $\rho = 1$, gets

$$Y(t)=\lim_{\substack{\rho\to1,\\ \tau\to t}} Y(t,\tau,\rho)$$

$$=A(t)e^{i\omega t}+\frac{k}{8\omega^2}A^3(t)e^{3i\omega t}-\frac{k^2A^5(t)}{64\omega^2}e^{5i\omega t}+\frac{3k^2A^4(t)\bar{A}(t)}{64\omega^2}e^{3i\omega t}$$

$$+\frac{qA(t)}{4\Omega}\left(\frac{e^{i(\omega+2\Omega)t}}{(\omega+\Omega)}+\frac{e^{i(\omega-2\Omega)t}}{(\omega-\Omega)}\right)+\frac{3kqA^2(t)\bar{A}(t)}{8\Omega^2\omega(\Omega^2-\omega^2)}\left[\frac{(\Omega-2\omega)}{(\omega+\Omega)}e^{i(\omega+2\Omega)t}+\frac{(\Omega+2\omega)}{(\Omega-\omega)}e^{i(\omega-2\Omega)t}\right]$$

$$+\frac{q^2A(t)}{32\Omega^2(\Omega^2-\omega^2)}\left[\frac{(\Omega-\omega)}{(\omega+2\Omega)}e^{i(\omega+4\Omega)t}+\frac{(\Omega+\omega)}{(\omega-2\Omega)}e^{i(\omega-4\Omega)t}\right]$$

$$-\frac{3kqA^3(t)}{16\Omega(\Omega^2-\omega^2)^2}\left[\frac{(\Omega-\omega)^2}{(2\omega+\Omega)}e^{i(3\omega+2\Omega)t}+\frac{(\Omega+\omega)^2}{(\Omega-2\omega)}e^{i(3\omega-2\Omega)t}\right]+cc. \tag{34}$$

5. Stability analysis at the non-resonance case

The stability criteria at the non-resonance case, can be obtained for solving equation (28). One may use the following polar form [1]:

$$A(\tau)=\tfrac{1}{2}\xi(\tau)e^{i\eta(\tau)}, \tag{35}$$

with real the unknowns functions $\xi(\tau)$ and $\eta(\tau)$. Insert (35) into the first-order solvability condition (27) which will separate into real and imaginary parts and gives

$$\xi(\tau)=\xi_0e^{-\frac{1}{2}\mu\tau}\quad\text{and}\quad \eta(\tau)=\frac{3k}{2\mu\omega}\xi_0e^{-\frac{1}{2}\mu\tau}+\eta_0, \tag{36}$$

where, ξ_0 and η_0 are integration constants. Clearly, the stability criteria at the non-resonance case requires that $\mu>0$.

5.1 The resonance case of Ω is near ω

Return to the first-order problem equation (27) and re-analyzed it in view of the nearness of Ω to ω. We express this approach by introducing the detuning parameter σ [1] such that

$$\Omega=\omega+\rho\sigma. \tag{37}$$

Accordingly, we have

$$-i(\omega-2\Omega)t=i\omega t+2i\sigma\tau. \tag{38}$$

Elimination of secular terms from equation (27), in view of (37) and (38) yields

$$A'+\tfrac{1}{2}\mu A-\frac{iq}{2\omega}\bar{A}e^{2i\sigma\tau}+\frac{3ik}{2\omega}A^2\bar{A}=0. \tag{39}$$

The first-order solution at this case is

$$Y_1(t,\tau)=\frac{q}{4\Omega(\omega+\Omega)}\left(Ae^{i(\omega+2\Omega)t}+\bar{A}e^{-i(\omega+2\Omega)t}\right)+\frac{k}{8\omega^2}\left(A^3e^{3i\omega t}+\bar{A}^3e^{-3i\omega t}\right) \tag{40}$$

Using (40) with equation (26) we obtain the uniform solution for the second-order problem, and the following solvability is presented:

$$A''+\mu A'+\frac{q^2}{4\Omega(\omega+\Omega)}A+\frac{kq}{8\omega^2}A^3e^{-2i\sigma\tau}-\frac{3kq}{4\Omega(\omega+\Omega)}A\bar{A}^2e^{2i\sigma\tau}-\frac{3k^2}{8\omega^2}A^3\bar{A}^2=0, \tag{41}$$

with its complex conjugate. The valid function $Y_2(t,\tau)$ is given by

$$Y_2(t,\tau)=\frac{3ik}{64\omega^3}\left(2A'+\mu A+2ikA^2\bar{A}\right)A^2e^{3i\omega t}-\frac{3k^2}{192\omega^4}A^5e^{5i\omega t}+\frac{q^2}{32\Omega^2(\omega+\Omega)(\omega+2\Omega)}Ae^{i(\omega+4\Omega)t}$$
$$+i\frac{q}{8\Omega^2(\omega+\Omega)^2}\left[\left(A'+\tfrac{1}{2}\mu A\right)(\omega+2\Omega)+3ikA^2\bar{A}\right]e^{i(\omega+2\Omega)t} \tag{42}$$
$$-\frac{kq}{16(2\omega+\Omega)(\omega+\Omega)}\left[\frac{3}{\Omega(\omega+\Omega)}-\frac{1}{2\omega^2}\right]A^3e^{i(3\omega+2\Omega)t}+cc.$$

The approximate solution up to the second-order is formulated by substituting from (14), (15), (40) and (42) into (15) gets

$$Y(t)=\lim_{\substack{\rho\to 1\\ \tau\to t}}\left(Y_0+\rho Y_1+\rho^2Y_2\right)$$
$$=Ae^{i\omega t}+\bar{A}e^{-i\omega t}+\frac{k}{8\omega^2}\left[A+\frac{3i}{8\omega}\left(2A'+\mu A+2ikA^2\bar{A}\right)\right]A^2e^{3i\omega t}-\frac{3k^2}{192\omega^4}A^5e^{5i\omega t}$$
$$+\frac{q}{4\Omega(\omega+\Omega)}\left\{A+i\frac{q}{2\Omega(\omega+\Omega)}\left[\left(A'+\tfrac{1}{2}\mu A\right)(\omega+2\Omega)+3ikA^2\bar{A}\right]\right\}e^{i(\omega+2\Omega)t}$$
$$+\frac{q^2}{32\Omega^2(\omega+\Omega)(\omega+2\Omega)}Ae^{i(\omega+4\Omega)t}-\frac{kq}{16(2\omega+\Omega)(\omega+\Omega)}\left[\frac{3}{\Omega(\omega+\Omega)}-\frac{1}{2\omega^2}\right]A^3e^{i(3\omega+2\Omega)t}+cc. \tag{43}$$

5.2 Stability analysis for the linear Mathieu equation

In the limiting case as $k\to 0$ into equation (1), linear damping Mathieu equation arrived. In this case the two solvability conditions (39) and (41) that produced at the resonance case of Ω is near ω having the following limit case

$$A'+\tfrac{1}{2}\mu A-\frac{iq}{2\omega}\bar{A}e^{2i\sigma\tau}=0, \tag{44}$$

$$A''+\mu A'+\frac{q^2}{4\Omega(\omega+\Omega)}A=0. \tag{45}$$

The first-order solvability condition (44) can be used to find the stability picture at the resonance case. The second-order solvability condition (45) can be used to find the value of the detuning parameter σ.

It is easy to show that equation (44) can be satisfied by the form

$$A(\tau)=\left[\left(\sigma+\frac{q}{2\omega}\right)\sin\Theta\tau+i\Theta\cos\Theta\tau\right]e^{\left(i\sigma-\frac{1}{2}\mu\right)\tau}, \tag{46}$$

where, the parameter μ must be positive, in order to find damping solution. The argument Θ is given by the following characteristic equation:

$$\Theta^2=\sigma^2-\frac{q^2}{4\omega^2}. \tag{47}$$

The parameter σ can be evaluated by substituting (46) into the second-order solvability condition (45) to gets

$$\sigma = -\frac{q}{2\omega} \; or \; \sigma = \pm \frac{\omega}{4q}\left(\mu^2 - \frac{q^2}{\omega^2} - \frac{q^2}{\Omega(\Omega+\omega)}\right). \tag{48}$$

The use of the first value of σ with (47) yields a zero solution for equation (44). For non-zero solution, the other values for σ are conforms. Inserting (48) into (47) gets

$$\Theta^2 = \frac{\omega^2}{16q^2}\left(\mu^2 - \frac{q^2}{\omega^2} - \frac{q^2}{\Omega(\Omega+\omega)}\right)^2 - \frac{q^2}{4\omega^2}. \tag{49}$$

Clearly, the stability criteria requires that the right-hand-side of (49) be positive, which implies that

$$\left(\mu^2 - \frac{q^2}{\omega^2} - \frac{q^2}{\Omega(\Omega+\omega)}\right)^2 - \frac{4q^4}{\omega^4} > 0. \tag{50}$$

Stability condition (50) can be rearranged in powers of the applied frequency Ω as

$$\Omega^2\left(\mu^2\omega^2 - 3q^2\right) + \Omega\omega\left(\mu^2\omega^2 - 3q^2\right) - q^2\omega^2 > 0\,, \tag{51}$$

and

$$\Omega^2\left(\mu^2\omega^2 + q^2\right) + \Omega\omega\left(\mu^2\omega^2 + q^2\right) - q^2\omega^2 < 0. \tag{52}$$

The transition curves separating stable state from unstable state corresponding to

$$\Omega_1 = \frac{-\omega\left(\mu^2\omega^2 - 3q^2\right) - \omega\sqrt{\mu^4\omega^4 - 2\mu^2\omega^2q^2 - 3q^4}}{2\left(\mu^2\omega^2 - 3q^2\right)}, \tag{53}$$

and

$$\Omega_2 = \frac{-\omega\left(\mu^2\omega^2 + q^2\right) - \omega\sqrt{\left(\mu^2\omega^2 + q^2\right)\left(\mu^2\omega^2 + 5q^2\right)}}{2\left(\mu^2\omega^2 + q^2\right)}. \tag{54}$$

5.3 Stability analysis for the nonlinear case

The first-order solvability condition (39) can be used to find the stability picture at the resonance case. The second-order solvability condition (41) can be used to find the value of the detuning parameter σ.

In order to relax the periodic term into equation (39) we let

$$A(\tau) = [\alpha(\tau) + i\beta(\tau)]e^{i\sigma\tau} \tag{55}$$

with real functions α and β. Insert (55) into (39), separating real and imaginary parts yields

$$\alpha' + \tfrac{1}{2}\mu\alpha - \left(\sigma + \frac{q}{2\omega} + \frac{3k}{2\omega}\left(\alpha^2 + \beta^2\right)\right)\beta = 0, \tag{56}$$

$$\beta' + \tfrac{1}{2}\mu\beta + \left(\sigma - \frac{q}{2\omega} - \frac{3k}{2\omega}\left(\alpha^2 + \beta^2\right)\right)\alpha = 0. \tag{57}$$

In order to solve the above coupled nonlinear equations (56) and (57), we may discuss the behavior at the steady-state response. This case is corresponding to the case of $\frac{d..}{d\tau} = 0$. If the solutions of (56) and (57), at the steady-state, are represented by α_0 and β_0, which are given by

$$\tfrac{1}{2}\mu\alpha_0-\left(\sigma+\frac{q}{2\omega}+\frac{3kr^2}{2\omega}\right)\beta_0=0, \tag{58}$$

$$\tfrac{1}{2}\mu\beta_0+\left(\sigma-\frac{q}{2\omega}-\frac{3kr^2}{2\omega}\right)\alpha_0=0, \tag{59}$$

where $r^2=\alpha_0^2+\beta_0^2$ is used. Equations (58) and (59) are two coupled algebraic equations in α_0 and β_0. For nontrivial solutions in α_0 and β_0, we obtain

$$\sigma^2=\frac{\left(q+3kr^2\right)^2}{4\omega^2}-\tfrac{1}{4}\mu^2. \tag{60}$$

In addition the constants α_0 and β_0 may be chosen as

$$\alpha_0=\left(\sigma+\frac{q+3kr^2}{2\omega}\right),\quad \text{and}\quad \beta_0=\tfrac{1}{2}\mu. \tag{61}$$

Squaring both equations in (61) and adding we get

$$\left(\sigma+\frac{q+3kr^2}{2\omega}\right)^2=r^2-\tfrac{1}{4}\mu^2. \tag{62}$$

Combing (60) with (62) yields

$$\sigma=\frac{2\omega^2r^2-\left(q+3kr^2\right)^2}{2\omega\left(q+3kr^2\right)}. \tag{63}$$

In order to find a constrain for bounded solution we may modulate the functions α and β as

$$\alpha(\tau)=\alpha_0+\alpha_1(\tau)\qquad \text{and}\qquad \beta(\tau)=\beta_0+\beta_1(\tau), \tag{64}$$

where the functions $\alpha_1(\tau)$ and $\beta_1(\tau)$ refer to a small deviation from the steady-state solution α_0 and β_0 . Then the system of (56) and (57) in the linearizing form becomes

$$\alpha_1'+\left(\tfrac{1}{2}\mu-\frac{3k}{\omega}\alpha_0\beta_0\right)\alpha_1-\left(\sigma+\frac{q+3kr^2}{2\omega}+\frac{3k}{\omega}\beta_0^2\right)\beta_1=0, \tag{65}$$

$$\beta'+\left(\tfrac{1}{2}\mu-\frac{3k}{\omega}\alpha_0\beta_0\right)\beta_1+\left(\sigma-\frac{q+3kr^2}{2\omega}-\frac{3k}{\omega}\alpha_0^2\right)\alpha_1=0. \tag{66}$$

The above system are two coupled linear differential equations of first-order in the two functions α_1 and β_1. This system can be satisfied by

$$\alpha_1(\tau)=\left(\sigma+\frac{q+3kr^2}{2\omega}+\frac{3k}{\omega}\beta_0^2\right)\, e^{-\left(\frac{1}{2}\mu-\frac{3k}{\omega}\alpha_0\beta_0\right)\tau}\,\sin\Theta\tau, \tag{67}$$

$$\beta_1(\tau)=e^{-\left(\frac{1}{2}\mu-\frac{3k}{\omega}\alpha_0\beta_0\right)\tau}\,\Theta\cos\Theta\tau, \tag{68}$$

where Θ is given by the following characteristic equation:

$$\Theta^2=\left(\sigma-\frac{q+9kr^2}{2\omega}+\frac{3k}{4\omega}\mu^2\right)\left(\sigma+\frac{q+3kr^2}{2\omega}+\frac{3k}{4\omega}\mu^2\right). \tag{69}$$

where relations (61) are used. This characteristic equation depends on the two related parameters σ and r^2. This relation between them is given in (60) or in (63).

By help of the second-order solvability condition (41) one can find an expression for both the unknowns σ and r^2 in terms of the frequency Ω. To accomplish this, one may substitute the steady-state solution

$$A(\tau)=(\alpha_0+i\beta_0)e^{i\sigma\tau}, \tag{70}$$

into the second-order solvability condition (41). Separating the real and imaginary parts, produces the following relations, between the parameters σ, Ω and r^2:

$$\sigma^2-\frac{q^2}{4\Omega(\omega+\Omega)}+\frac{1}{16}kq\mu^2\left(\frac{\Omega^2+\omega\Omega-6\omega^2}{\omega^2\Omega(\omega+\Omega)}\right)-\left[\frac{kq}{8}\left(\frac{\Omega^2+\omega\Omega-6\omega^2}{\omega^2\Omega(\omega+\Omega)}\right)-\frac{3k^2}{8\omega^2}\left(\sigma+\frac{q+3kr^2}{2\omega}\right)\right]r^2=0, \tag{71}$$

$$\sigma+\frac{kq}{8}\left(\frac{\Omega^2+\omega\Omega+6\omega^2}{\omega^2\Omega(\omega+\Omega)}\right)\left(\sigma+\frac{q+3kr^2}{2\omega}\right)+\frac{3k^2}{2\omega^2}r^2=0, \tag{72}$$

where relations (61) are used. Removing the parameter σ from (72), by using its equivalent in (63), gives a polynomial of second-order in r^2

$$r^4+\frac{\Omega(\omega+\Omega)(12k^2q+8\omega^3-23kq\omega)+6kq\omega^3}{36k^2(k-\omega)\Omega(\omega+\Omega)}r^2-\frac{q^2\omega}{9k^2(k-\omega)}=0;\ k\neq\omega \tag{73}$$

Replacing r^4 and r^2 into (71) with their equivalents in (63) and (72) leads to the following quadratic equation in the detuning parameter σ:

$$\begin{aligned}&6k\omega\Omega(\omega+\Omega)(8\omega^2-1)\left[8\omega\Omega(\omega+\Omega)+kq(\Omega^2+\omega\Omega+6\omega^2)\right]\sigma^2\\&-\left[\begin{aligned}&\Omega^2(\omega+\Omega)^2\left[2\omega(8\omega+kq)(3k^2q-3kq+4\omega^2-2\omega q)+3k^2q^2\right]\\&+6\omega^2\Omega(\omega+\Omega)\left[3k^2q^2+8\omega^4+4kq^2\omega^2+2kq\omega(3k^2q-3kq+4\omega^2-2\omega q)\right]+144kq^2\omega^6\end{aligned}\right]\sigma\\&+\Omega^2(\omega+\Omega)^2\left[3k^2q(\omega\mu^2-q)(8\omega+kq)+kq^2(3kq-4\omega^2+2\omega q)\right]-36k\omega^5(3k^2\mu^2+2kq+2q^3)\\&+6k\omega^2\Omega(\omega+\Omega)\left[3k^2q^2(\omega\mu^2-q)-q\omega(3k\mu^2+2q)(8\omega+kq)+q^2(3kq-4\omega^2+2\omega q)-2q^2\omega\right]=0.\end{aligned} \tag{74}$$

This equation gives two values σ_1 and σ_2 for the detuning parameter σ which makes the solution (55) without unknowns.

Clearly, the stabilization for the problem requires that the right-hand side of (69) be positive provided that the exponential in (67) and (68) has positive values. It is noted that the stability reveals as the coefficient of the periodic term in (1) tends to zero. The instability arrived as the parameter q going away the zero value. Thus, the stability conditions are found as

$$\mu>0,\quad \frac{3k}{\omega}\sigma+\frac{3kq+9k^2r^2}{2\omega^2}-1<0, \tag{75}$$

$$\sigma > \frac{q+6kr^2}{2\omega} - \frac{3k}{4\omega}\mu^2, \tag{76}$$

$$\sigma + \frac{q+3kr^2}{2\omega} + \frac{3k}{4\omega}\mu^2 < 0. \tag{77}$$

Removing the parameter r^2 from the above stability conditions by using (72) yields the following two conditions for stability:

$$\mu > 0, \qquad \omega(k-1)\sigma < \frac{1}{6}\left(2\omega^2 - 3kq\right) + \frac{2kq^2\left(\Omega^2+\omega\Omega+6\omega^2\right)}{8\omega\Omega(\omega+\Omega)+kq\left(\Omega^2+\omega\Omega+6\omega^2\right)}, \tag{78}$$

$$\frac{3}{k}\sigma > \frac{q}{\omega} - \frac{3q^2\left(\Omega^2+\omega\Omega+6\omega^2\right)}{2\omega\left[8\omega\Omega(\omega+\Omega)+kq\left(\Omega^2+\omega\Omega+6\omega^2\right)\right]}. \tag{79}$$

The transition curves separating stable state from unstable one are corresponding to

$$\sigma = \frac{\left(2\omega^2-3kq\right)\left[8\omega\Omega(\omega+\Omega)+kq\left(\Omega^2+\omega\Omega+6\omega^2\right)\right]+12kq^2\left(\Omega^2+\omega\Omega+6\omega^2\right)}{6\omega(k-1)\left[8\omega\Omega(\omega+\Omega)+kq\left(\Omega^2+\omega\Omega+6\omega^2\right)\right]}, \text{ and} \tag{80}$$

$$\sigma = \frac{2kq\left[8\omega\Omega(\omega+\Omega)+kq\left(\Omega^2+\omega\Omega+6\omega^2\right)\right]-3kq^2\left(\Omega^2+\omega\Omega+6\omega^2\right)}{6\omega\left[8\omega\Omega(\omega+\Omega)+kq\left(\Omega^2+\omega\Omega+6\omega^2\right)\right]}. \tag{81}$$

Using the definition (37) the above transition curves can be sought within the parameter ρ

$$\Omega = \omega + \rho\frac{\left(2\omega^2-3kq\right)\left[8\omega\Omega(\omega+\Omega)+kq\left(\Omega^2+\omega\Omega+6\omega^2\right)\right]+12kq^2\left(\Omega^2+\omega\Omega+6\omega^2\right)}{6\omega(k-1)\left[8\omega\Omega(\omega+\Omega)+kq\left(\Omega^2+\omega\Omega+6\omega^2\right)\right]}, \tag{82}$$

$$\Omega = \omega + \rho\frac{2kq\left[8\omega\Omega(\omega+\Omega)+kq\left(\Omega^2+\omega\Omega+6\omega^2\right)\right]-3kq^2\left(\Omega^2+\omega\Omega+6\omega^2\right)}{6\omega\left[8\omega\Omega(\omega+\Omega)+kq\left(\Omega^2+\omega\Omega+6\omega^2\right)\right]}. \tag{83}$$

In order to obtain the transition curves independent of the parameter ρ, we may inserting (80) as well as (81), into the relation (74) , then the following transition curves are imposed:

$$a_3\Omega^3(\Omega+\omega)^3 + a_2\omega^2\Omega^2(\Omega+\omega)^2 + 36a_1\omega^4\Omega(\Omega+\omega) + a_0 = 0, \tag{84}$$

$$b_3\Omega^3(\Omega+\omega)^3 + b_2\omega^2\Omega^2(\Omega+\omega)^2 + 36b_1\omega^4\Omega(\Omega+\omega) + b_0 = 0. \tag{85}$$

It is noted that the instability state lies between the above transition curves. The constant coefficients a_j and b_j, $j = 0,1,2,3$ are given below:

$$\begin{aligned} a_3 &= k\left(8\omega^2-1\right)\left[\left(2\omega^2-3kq\right)(8\omega+kq)+12kq^2\right]^2 \\ &\quad -(k-1)\left[\left(2\omega^2-3kq\right)(8\omega+kq)+12kq^2\right]\left[2\omega(8\omega+kq)\left(3k^2q-3kq+4\omega^2-2\omega q\right)+3k^2q^2\right] \\ &\quad +6\omega(k-1)^2(8\omega+kq)\left[3k^2q\left(\omega\mu^2-q\right)(8\omega+kq)+kq^2\left(3kq-4\omega^2+2\omega q\right)\right], \end{aligned}$$

$$a_2 = 12k^2q(8\omega^2-1)(2\omega^2-3kq+12q)[(2\omega^2-3kq)(8\omega+kq)+12kq^2]$$
$$-6(k-1)[(2\omega^2-3kq)(8\omega+kq)+12kq^2][3k^2q^2+8\omega^4+4kq^2\omega^2+2kq\omega(3k^2q-3kq+4\omega^2-2\omega q)]$$
$$-6kq(k-1)(2\omega^2-3kq+12q)[2\omega(8\omega+kq)(3k^2q-3kq+4\omega^2-2\omega q)+3k^2q^2]$$
$$+6\times36k\omega(k-1)^2k^2q^2(\omega\mu^2-q)(8\omega+kq)+36k\omega(k-1)^2q^2(3kq-4\omega^2+2\omega q)(8\omega+2kq)$$
$$-36k\omega(k-1)^2q\omega(3k\mu^2+2q)(8\omega+kq)^2-72k\omega(k-1)^2q^2\omega(8\omega+kq),$$

$$a_1 = +k^3q^2(8\omega^2-1)(2\omega^2-3kq+12q)^2-4kq^2\omega^2(k-1)[(2\omega^2-3kq)(8\omega+kq)+12kq^2]$$
$$-kq(k-1)(2\omega^2-3kq+12q)[3k^2q^2+8\omega^4+4kq^2\omega^2+2kq\omega(3k^2q-3kq+4\omega^2-2\omega q)]$$
$$+6k^2q\omega(k-1)^2[3k^2q^2(\omega\mu^2-q)-q\omega(3k\mu^2+2q)(8\omega+kq)+q^2(3kq-4\omega^2+2\omega q)-2q^2\omega]$$
$$-6k\omega^2(k-1)^2(8\omega+kq)(3k^2\mu^2+2kq+2q^3),$$

$$a_0 = -144kq^2\omega^5 6kq\omega^3(k-1)(2\omega^2-3kq+12q)-36\times36k^2q\omega^8(k-1)^2(3k^2\mu^2+2kq+2q^3),$$

$$b_3 = k^2q^2(8\omega^2-1)[2(8\omega+kq)-3q]^2+6q\omega(8\omega+kq)[3k(\omega\mu^2-q)(8\omega+kq)+q(3kq-4\omega^2+2\omega q)]$$
$$-[2q(8\omega+kq)-3q^2][2\omega(8\omega+kq)(3k^2q-3kq+4\omega^2-2\omega q)+3k^2q^2],$$

$$b_2 = 12k^2q^3(8\omega^2-1)(2k-3)[2(8\omega+kq)-3q]+6kq^2[3k(\omega\mu^2-q)(8\omega+kq)+q(3kq-4\omega^2+2\omega q)]$$
$$-6q[2(8\omega+kq)-3q][3k^2q^2+8\omega^4+4kq^2\omega^2+2kq\omega(3k^2q-3kq+4\omega^2-2\omega q)]$$
$$-6q^2(2k-3)[2\omega(8\omega+kq)(3k^2q-3kq+4\omega^2-2\omega q)+3k^2q^2]$$
$$+36\omega(8\omega+kq)[3k^2q^2(\omega\mu^2-q)-q\omega(3k\mu^2+2q)(8\omega+kq)+q^2(3kq-4\omega^2+2\omega q)-2q^2\omega],$$

$$b_1 = k^2q^4(8\omega^2-1)(2k-3)^2-4q^2\omega^2[2kq(8\omega+kq)-3kq^2]-6\omega^2(3k^2\mu^2+2kq+2q^3)(8\omega+kq)$$
$$-q(2k-3)[3k^2q^2+8\omega^4+4kq^2\omega^2+2kq\omega(3k^2q-3kq+4\omega^2-2\omega q)]$$
$$+6kq\omega[3k^2q^2(\omega\mu^2-q)-q\omega(3k\mu^2+2q)(8\omega+kq)+q^2(3kq-4\omega^2+2\omega q)-2q^2\omega],$$

$$b_0 = -72\times6\omega^8kq(9k^2\mu^2+6kq+4kq^3).$$

6. Conclusion

In this study we proposes a variation of the homotopy perturbation method, by using a modulation technique, which allows to find solutions for ordinary differential equations with periodic coefficients. This work has been employed to analyze of parametrically excited oscillator without smallness the cubic nonlinearity. The simplest equation of this type is the Mathieu equation which usually contains a small parameter [14,15]. As in the homotopy perturbation [3], the analysis has no dependence on equations having a small parameter. Due to this modulation technique a solvability condition at each level of perturbation is imposed. Solving these solvability conditions leads to studying the stability behavior. Stability conditions, in both resonance and non-resonance cases are derived.

References

[1] Nayfeh Ali.H, Mook D.T., "Non-Linear Oscillations, John Wily & Sons, New York (1979).

[2] Nash C, Sen S., "Topology and Geometry for physicists," Academic Press. rnc, London (1983).
[3] J.H. He, Homotopy perturbation technique, Comp. Meth. Appl. Mech.Eng. 178, 1999, 257-262.
[4] J.H. He, A coupling method of homotopy technique and a perturbation technique for non-linear problems, Int. J. Non-Linear. Mech. 35, 2000, 37-43.
[5] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos Solitons Fractals. 26, 2005, 295-300.
[6] J.H. He, Homotopy perturbation method for solving boundary value problems, Phys. Lett. A 350, 2006, 87-88
[7] J.H. He, Homotopy perturbation method: A new nonlinear analytical technique, Appl. Math. Comput. 135, 2003, 73-79.
[8] J.H. He, A simple perturbation approach to Blasius equation, Appl. Math. Comput. 140, 2003, 217-222.
[9] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos Solitons Fractals. 26, 2005, 295-300.
[10] J.H. He, Homotopy perturbation method for solving boundary value problems, Phys. Lett. A 350, 2006, 87-88.
[11] E. Babolian, J. Saeidian, and A. Azizi, Applied Mathematical Sciences, Vol. 3, 2009, no. 45, 2215 – 2226.
[12] M. Rabbani, Journal of mathematics and computer Science, 7, 2013, 272 – 275.
[13] Z.Ayati, J. Biazar and S. Ebrahimi, Journal of Interpolation and Approximation in Scientific Computing, 2014, 1-8.
[14] Y.O. El-Dib, Chaos, Solitons and Fractals 12, 2001, 705-720.
[15] Y.O. El-Dib, Science and Engineering Applications 1(5), 2016, 76-88.